J. Fluid Mech. (1967), *vol.* 27, *part* 3, *pp.* 465–492
Printed in Great Britain

Finite-amplitude instability of parallel shear flows

By W. C. REYNOLDS
Department of Mechanical Engineering, Stanford University
AND MERLE C. POTTER
Mechanical Engineering Department, Michigan State University

(Received 13 September 1965 and in revised form 29 March 1966)

A formal expansion method for analysis of the non-linear development of an oblique wave in a parallel flow is presented. The present approach constitutes an extension and modification of the method of Stuart and Watson. Results are obtained for plane Poiseuille flow, and for a combination of plane Poiseuille and plane Couette flow. The Poiseuille flow exhibits finite-amplitude subcritical instability, and relatively weak but finite disturbances markedly reduce the critical Reynolds number. The combined flow, which becomes stable to infinitesimal disturbances at all Reynolds numbers when the Couette component is sufficiently great, remains unstable to finite disturbances.

1. Introduction

Two important questions arise in the consideration of non-linear aspects in hydrodynamic stability theory for parallel flows. The first, which prompted the present work, is whether or not a flow which is stable to infinitesimal disturbances might be unstable to disturbances of some finite amplitude. This is particularly important in the case of plane Couette flow, which appears to be stable to infinitesimal disturbances at all the Reynolds numbers. Secondly, one can seek the nature of the finite-amplitude equilibrium flow which develops as a result of an initial instability. The means for answering these questions are due primarily to the work of J. T. Stuart and his associates; the central physical aspects have been summarized by Stuart (1960*b*), and the key mathematical ideas given by Stuart (1960*a*) and Watson (1960, 1962).

The non-linear analyses centre about an equation for the amplitude of the velocity disturbance of the form

$$dA/dt = a^{(0)}A + a^{(2)}A^3 + \dots \qquad (1.1)$$

A classical linearized stability analysis yields the constant $a^{(0)}$ as an eigenvalue of the Orr–Sommerfeld problem.† The aim of the non-linear theory is to determine the remaining $a^{(n)}$, and particularly $a^{(2)}$. If $a^{(0)} < 0$ the flow is stable to small disturbances, but the question remains as to whether disturbances of sufficient amplitude might produce 'subcritical' instability. This would be reflected by the contribution of the higher-order terms outweighing the $a^{(0)}$ term. In contrast, if

† $a^{(0)} = \alpha c_i$, where c is the eigenvalue of linearized stability theory (Lin 1955).

$a^{(0)} > 0$ the flow is unstable to small disturbances; but it is possible that the higher-order terms may balance the leading term, and a finite-amplitude supercritical equilibrium flow can thereby be obtained. In the special case $a^{(0)} = 0$, corresponding to the neutral stability curve of linearized theory, the sign of $a^{(2)}$ determines whether the disturbances actually grow or decay. The general behaviour of the amplitude, as given by an equation of the type (1.1), can be conveniently represented in the phase plane (figure 1).

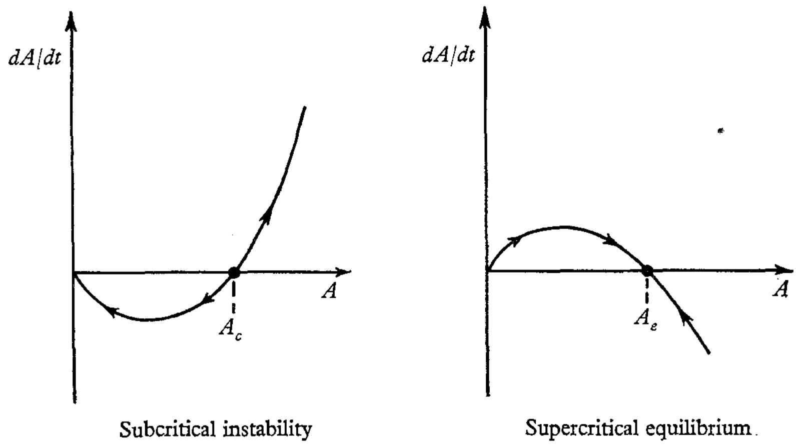


FIGURE 1. The amplitude phase plane.

Perhaps the most well-known case of a supercritical equilibrium flow is the Taylor vortex flow between rotating cylinders. In fact, several modes of supercritical equilibrium are known (Coles 1965). Davey (1962) calculated $a^{(2)}$ for this case, using an extension of Stuart's (1960*a*) formalism, and obtained remarkably good agreement with experimentally measured torques in the range of the first supercritical equilibrium flow. DiPrima & Stuart (1964) have further extended the ideas to treat the instability of the first supercritical equilibrium flow. Coles (1965) has shown experimentally how turbulent flow is eventually obtained following a complex spectral evolution. Thus, a great deal is now known about the nature of the non-linear interactions in rotating Couette flow, and the methods of analysis proposed by Stuart have shown their value.

The initial instability in the rotating case is due primarily to centrifugal effects. In contrast, the initial instability in plane Poiseuille flow is due to rather subtle viscous effects (Lin 1955). Moreover, there has been no experimental evidence suggesting the existence of any supercritical equilibrium flows for this case, and the initial instability seems to lead catastrophically to turbulence. Furthermore, there is some evidence that subcritical instabilities exist in this type of parallel shear flow (Meksyn & Stuart 1951; Davies & White 1928). Hence it is likely that $a^{(2)}$ is *positive* for plane Poiseuille flow, at least in the region of the critical Reynolds number. However, an approximate integral method treatment of the non-linear problem by Stuart (1958) gave a conflicting result; $a^{(2)}$ was found to be *negative*, indicating the existence of supercritical equilibrium states, and no subcritical instabilities. Stuart's (1960*a*) work on the expansion method was motivated largely by a desire to resolve this controversy, and for the present paper the required numerical work has been completed.

A third flow of concern is plane Couette flow, which is believed to be stable to infinitesimal disturbances at all the Reynolds numbers. It is likely that finite-amplitude disturbances in this flow may grow, but as yet this question remains unanswered. Certain aspects of the Stuart–Watson expansion formalism require modification for treatment of flows which are highly stable to small disturbances, and it was primarily the desire to examine a special case of the plane Couette flow problem that prompted the present work on the expansion method. The numerical work associated with this case will be described in a subsequent paper.

In the problem mentioned above, three-dimensional finite amplitude instabilities were of particular interest. While the initial Stuart–Watson approach was developed only for two-dimensional disturbances, in a subsequent paper Stuart (1961) considered the interaction of a two-dimensional disturbance with a particular three-dimensional wave. The present expansion method is essentially a recasting of the Stuart–Watson approach, extended to include a class of three-dimensional disturbances. Stuart and Watson felt that their method was limited to cases where the rate of growth or decay of infinitesimal disturbances is very small; a modification suggested in the present work may be more useful in handling equilibrium flows.

In a two-dimensional *linear* stability analysis, a key assumption is that the perturbation stream function, ψ', representing the departure of the flow field from the basic steady, parallel laminar flow, can be represented by harmonic components, any one of which is of the form $\psi' = 2\,\mathrm{Re}\{\phi(y)\exp[i\alpha(x-ct)]\}$. In a linearized analysis these harmonics do not interact, and hence their behaviour can be considered independently. If assumed stream function perturbations of this form are substituted into the equations of motion, and only the first-order terms retained, one obtains the well-known Orr–Sommerfeld problem for the disturbance eigenfunction $\phi(y)$,

$$\left.\begin{aligned}\{(D^2-\alpha^2)^2 - i\alpha R[(\overline{u}-c)(D^2-\alpha^2)-D^2\overline{u}]\}\phi = 0,\\ \phi = D\phi = 0 \quad \text{at} \quad y = y_1, y_2.\end{aligned}\right\} \tag{1.2}$$

Here y_1 and y_2 represent the bounds of the flow, $D = d/dy$, $\overline{u}(y)$ is the basic parallel flow velocity, and R is the flow Reynolds number (Lin 1955). The complex constant $c = c_r + ic_i$ becomes the eigenvalue of the problem; c_r represents the speed at which a wave propagates downstream,† and c_i characterizes the rate at which the disturbance grows or decays in time.‡ The shape of the disturbance is determined by the eigenfunction $\phi(y)$, which in turn depends on the prescribed values of the wave-number (α) and R.

In a non-linear analysis one is led naturally to expanding the stream function in terms of harmonics of the basic Orr–Sommerfeld wave, but this must be done with some caution. The non-linearity can be expected to have three important effects on the motion. First, interaction of the basic wave with itself produces a mean 'Reynolds stress', which in turn distorts the mean velocity field. Secondly,

† Here we consider real α only.
‡ $\alpha c_i = a^{(0)}$.

the amplitude of an unstable disturbance grows until the mean field has again become stable, and in a sense the non-linearity therefore limits the ultimate amplitude. Thirdly, it may be expected that the non-linearity will modify the wave speed. Any expansion or perturbation method which is to be successful must allow for all three of these effects, and in particular for the dependence of the wave speed on amplitude. The approach of Stuart and Watson does this, but in a somewhat disguised way. An intent of the present formalism is to bring this third effect more clearly into view.

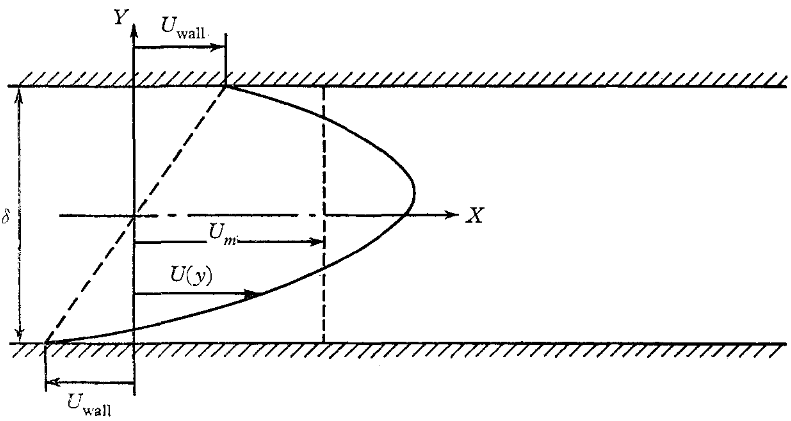


FIGURE 2. Co-ordinates and normalization for the two problems. $R = U_m\delta/\nu$, $y = Y/\delta$. Dimensionless velocity profile: $u = \frac{3}{2}(1-y^2)+u_w y$, $u_w = U_{wall}/U_m$.

The expansion formalism will be developed in a general way; we have carried out the detailed calculations for plane Poiseuille flow, and for a combination of plane Poiseuille and plane Couette flow. The latter affords a unique opportunity to examine the finite-amplitude stability of a flow which is stable at all Reynolds numbers to infinitesimal disturbances.

The co-ordinate system and non-dimensionalization scheme appropriate in these two problems are indicated in figure 2. Note that the bulk average velocity has been used in the normalization, rather than the centre-line velocity used by Lin (1955) and others. This choice permits the total flow to be maintained constant in the non-linear problem.

2. The expansion formalism

We begin with the Navier–Stokes equations in a suitably normalized form; using the usual subscript summation convention, they are

Momentum
$$\frac{\partial u_i}{\partial t}+u_j\frac{\partial u_i}{\partial x_j}+\frac{\partial p}{\partial x_i}-\frac{1}{R}\frac{\partial^2 u_i}{\partial x_j \partial x_j}=0, \tag{2.1a}$$

Continuity
$$\partial u_i/\partial x_i = 0. \tag{2.1b}$$

The variables are presumed to be normalized on suitable characteristic lengths and velocities, which remain constant in time (see figure 2).

In a linearized analysis of three-dimensional disturbances we would assume velocity perturbations of the form

$$u_1' = \hat{u}_i(x_2)\exp\left[i(\alpha x_1+\beta x_3+\omega t)\right]\exp(at)$$

and $\omega - ia$ would emerge as the (complex) eigenvalue. The term $\exp(at)$ would represent the disturbance amplitude, and the stability would be determined by the sign of a. In the non-linear analysis we seek a solution in terms of this basic wave and its harmonics, and hence an initial transformation of variables is suggested. We put

$$\theta = \alpha x_1+\beta x_3+\omega t, \quad \omega = \omega(A), \quad A = A(t), \quad y = x_2. \tag{2.2}$$

The constants α and β represent the streamwise (x_1) and transverse (x_3) wave-numbers, and ω is the frequency of the basic wave. Note that we let this frequency depend upon the amplitude $A(t)$, which remains to be defined in some suitable manner.

If we were considering two-dimensional motions, the transformation to (θ, A, y)-space would leave the number of independent variables unchanged; since time appears in two places in the new variables, the transformation is essentially an application of the 'method of two times'. When three-dimensional motions are considered, the transformation reduces the number of independent variables from four to three, and in a sense is analogous to the Squire transformation of linearized stability theory (cf. Lin 1955).

In terms of the new variables, (2.1) becomes

Momentum

$$\frac{dA}{dt}\frac{\partial u_i}{\partial A}+\left\{\omega+\frac{dA}{d\omega}\left(t\frac{dA}{dt}\right)+\alpha u_1+\beta u_3\right\}\frac{\partial u_i}{\partial \theta}+u_2\frac{\partial u_i}{\partial y}$$
$$+\left(\alpha\frac{\partial p}{\partial \theta},\frac{\partial p}{\partial y},\beta\frac{\partial p}{\partial \theta}\right)-\frac{1}{R}\left\{(\alpha^2+\beta^2)\frac{\partial^2 u_i}{\partial \theta^2}+\frac{\partial^2 u_i}{\partial y^2}\right\} = 0, \tag{2.3a}$$

Continuity

$$\frac{\partial(\alpha u_i+\beta u_3)}{\partial \theta}+\frac{\partial u_2}{\partial y} = 0. \tag{2.3b}$$

Note that we seek solutions in terms of the instantaneous position in the cycle of the basic wave (θ), the amplitude of the fluctuations (A), and the distance from the wall (y).

The form of (2.3b) suggests that we represent the problem in terms of a stream function, ψ, defined by

$$\partial\psi/\partial y = \alpha u_1+\beta u_3, \quad \partial\psi/\partial\theta = -u_2. \tag{2.4}$$

Now, if we multiply the u_1 momentum equation by α, the u_3 momentum equation by β, and add, an equation involving only ψ and p is obtained. The u_2 momentum equation may likewise be expressed in terms of ψ and p, and hence by appropriate cross-differentiation and combination p can be eliminated. We thereby obtain a fourth-order equation for ψ,

$$\frac{dA}{dt}\frac{\partial \zeta}{\partial A}+\left[\omega+\frac{d\omega}{dA}\left(t\frac{dA}{dt}\right)+\frac{\partial\psi}{\partial y}\right]\frac{\partial\zeta}{\partial\theta}-\frac{\partial\psi}{\partial\theta}\frac{\partial\zeta}{\partial y}-\frac{1}{R}\left(\frac{\partial^2\zeta}{\partial y^2}+\kappa^2\frac{\partial^2\zeta}{\partial\theta^2}\right) = 0, \tag{2.5a}$$

where for brevity we put

$$\zeta = \partial^2\psi/\partial y^2 + \kappa^2(\partial^2\psi/\partial\theta^2), \quad \kappa^2 = \alpha^2+\beta^2. \tag{2.5b, c}$$

Note the similarity of (2.5) and the vorticity equation for two-dimensional motions.

The three-dimensional motion would appear as a wave running obliquely to the main flow direction at an angle $\tan^{-1}(\beta/\alpha)$. The stream function represents the motion in planes which are perpendicular to the walls and aligned with the direction of propagation of this basic wave, and ζ is proportional to the vorticity of this motion.

For the combined flow, the boundary conditions on ψ are

$$\partial\psi/\partial\theta = 0, \quad \partial\psi/\partial y = \pm\alpha u_w \quad \text{at} \quad y = \pm 1, \tag{2.5d}$$

ψ periodic in θ with period 2π. As we shall shortly see, an initial condition is not necessary for the investigation of non-linear stability. The transformations above, which constitute some of the main differences from the Stuart–Watson formalism, have reduced the number of independent variables from four to one; but the most important gain is that the eigenvalue ω now appears explicitly in the differential equation, in a place where it can be expanded along with ψ in the formal treatment.

We next expand the stream function in terms of its harmonic components. Using the superscript summation convention (appendix A), we put

$$\psi(A, y, \theta) = \Psi^{(k)}(A, y)\, e^{ik\theta} + \tilde{\Psi}^{(k)}(A, y)\, e^{-ik\theta}. \tag{2.6}$$

Note that the convention requires summation over all positive integers k; the tilde denotes a complex conjugate. Substituting into (2.5), and separating out the coefficients of like exponentials, we obtain an infinite set of coupled non-linear partial differential equations for the harmonic amplitudes. The coefficient of $\exp(ik\theta)$ yields, without approximation,†

$$\begin{aligned} &\frac{dA}{dt}\frac{\partial Z^{(k)}}{\partial A} + \left[\omega + \frac{d\omega}{dA}\left(t\frac{dA}{dt}\right)\right] ikZ^{(k)} \\ &\quad + \frac{1}{1+\delta_{k0}}\left\{\frac{\partial\Psi^{(k-j)}}{\partial y}[ijZ^{(j)}] + \frac{\partial\tilde{\Psi}^{(j)}}{\partial y}[i(k+j)\, Z^{(k+j)}] - \frac{\partial\Psi^{(k+j)}}{\partial y}[ij\tilde{Z}^{(j)}]\right. \\ &\quad \left. - [i(k-j)\Psi^{(k-j)}]\frac{\partial Z^{(j)}}{\partial y} - [-ij\tilde{\Psi}^{(j)}]\frac{\partial Z^{(k+j)}}{\partial y} - [i(k+j)\Psi^{(k+j)}]\frac{\partial \tilde{Z}^{(j)}}{\partial y}\right\} \\ &\quad - \frac{1}{R}\left(\frac{\partial^2}{\partial y^2} - k^2\kappa^2\right) Z^{(k)} = 0, \end{aligned} \tag{2.7a}$$

where $\delta_{kj} = 0$ if $k \neq j$, $\delta_{kj} = 1$ if $k = j$, and

$$Z^{(k)} = [(\partial^2/\partial y^2) - k^2\kappa^2]\, \Psi^{(k)}. \tag{2.7b}$$

The boundary conditions $(2.5d)$ then give

$$\Psi^{[k]} = \partial\Psi^{[k]}/\partial y = 0 \quad \text{at} \quad y = y_1, y_2. \tag{2.7c}$$

The non-linearity and coupling of this infinite set makes its solution difficult. However, if the amplitude of the wave is small we can seek a solution as a power

† The terms are summed over the repeated superscript j, for all values of j for which *both* superscripts $\geqslant 0$.

series in the amplitude, A, and thereby obtain sufficient decoupling that a sequential solution becomes possible. We shall require the solution for infinitesimal amplitude to reduce to the Orr–Sommerfeld wave, and the solution for zero amplitude to reduce to the basic laminar flow. Hence the $O(A)$ terms must represent the Orr–Sommerfeld fundamental and the $O(1)$ terms the laminar flow. Upon further consideration of the non-linear interaction, we see that the $O(A^2)$ terms involve the second harmonic ($k = 2$) and additional mean terms ($k = 0$), both generated by interaction of the fundamental with itself. These in turn interact with the fundamental to produce $O(A^3)$ terms, containing the third harmonic ($k = 3$), and strengthening the fundamental ($k = 1$). These considerations suggest that we seek a solution in the form

$$\Psi^{(k)}(A, y) = A^n \phi^{(k;n)}(y). \tag{2.8}$$

Note that, under the superscript summation convention, (2.8) represents a sum over all $n \geqslant k$. Hence $\Psi^{(k)}$ contains no terms of order less than A^k, and it is this feature which will decouple the equations for $\phi^{(k;n)}$ to the point where they can be solved sequentially.

We must next decide how to handle the term dA/dt. Since we have assumed a power series expansion in A for $\Psi^{(k)}$, and since we want the amplitude for infinitesimal A to behave as in linear theory (exponentially), we put

$$A^{-1} dA/dt = a^{(0)} + Aa^{(1)} + A^2 a^{(2)} + \ldots = A^n a^{(n)}. \tag{2.9}$$

The $a^{(n)}$ are constants which must be determined; $a^{(0)}$ emerges as an eigenvalue from the linearized analysis, $a^{(1)}$ will turn out to be zero, and $a^{(2)}$ becomes the centre of interest in the non-linear problem. If the flow is neutrally stable to infinitesimal disturbances, then $a^{(0)} = 0$, and $a^{(2)}$ consequently determines whether a weak disturbance will grow or decay. We shall return to this important equation shortly.

Finally, we represent the term involving ω by a power series in A,

$$\omega + \frac{d\omega}{dt}\left(t\frac{dA}{dt}\right) = b^{(0)} + Ab^{(1)} + \ldots = A^n b^{(n)}. \tag{2.10}$$

This is essentially a Poincaré eigenvalue stretching, in terms of undetermined constants $b^{(n)}$. Note that $b^{(n)}$ represents the $O(A^n)$ contribution to the frequency of an equilibrium motion ($dA/dt = 0$); $b^{(0)}$ will emerge as an eigenvalue of the linearized theory, $b^{(1)}$ will turn out to be zero, and consequently $b^{(2)}$ will reflect the change in oscillation frequency by the non-linearity. With these two expansions, a set of sequentially coupled ordinary differential equations for the $\phi^{(k;n)}$ will be obtained. These will involve the $a^{(n)}$ and $b^{(n)}$, which must be determined in some appropriate manner. Once these are found, we can return to (2.9) and (2.10) to determine the amplitude and oscillation frequency as functions of time.

Substituting (2.8)–(2.10) into (2.7), and collecting the terms of various orders, we obtain an infinite set of equations for the $\phi^{(k;n)}$. The coefficient of A^n yields

$$\begin{aligned} &ma^{(n-m)} z^{(k;m)} + b^{(n-m)} ikz^{(k;m)} + [1/(1+\delta_{k0})] \{ D\phi^{(k-j;\,n-m)} [ijz^{(j;m)}] \\ &\quad + D\tilde{\phi}^{(j;\,n-m)} [i(k+j)\, z^{(k+j;\,m)}] + D\phi^{(k+j;\,n-m)} [-ij\tilde{z}^{(j;m)}] - [i(k-j)\, \phi^{(k-j;\,n-m)}] \\ &\quad \times Dz^{(j;m)} - [-ij\tilde{\phi}^{(j;\,n-m)}] Dz^{(k+j;\,m)} - [i(k+j)\, \phi^{(k+j;\,n-m)}] D\tilde{z}^{(j;m)} \} \\ &\quad - R^{-1}(D^2 - k^2\kappa^2) z^{(k;n)} = 0, \end{aligned} \tag{2.11a}$$

where $D = d/dy$, and

$$z^{(k;\,n)} = (D^2 - k^2\kappa^2)\,\phi^{(k;\,n)}. \tag{2.11b}$$

We next collect the terms involving $\phi^{(k;\,n)}$; the superscript convention is particularly helpful in factoring these out of the summed products. In addition, the basic laminar flow $\bar{u}(y)$ which is presumed known is related to $\phi^{(0;\,0)}$, the $O(A^0)$ contribution to the zeroth harmonic, by

$$\bar{u} = 2\alpha^{-1}D\phi^{(0;\,0)} = 2\alpha^{-1}D\tilde{\phi}^{(0;\,0)}. \tag{2.12}$$

When the terms involving $\phi^{(0;\,0)}$ are accordingly represented in terms of $\bar{u}$, the system (2.11) may be written as

$$L_{kn}\phi^{[k;\,n]} = i\alpha c^{[n-1]}G\delta_{k1} + H_{kn}. \tag{2.13a}$$

Here δ_{kj} is the Kronecker delta; the operator L_{kn} is

$$L_{kn} = ik[(-i(n/k)\,a^{(0)} + b^{(0)} + \alpha\bar{u})\,(D^2 - k^2\kappa^2) - \alpha(D^2\bar{u})] - R^{-1}(D^2 - k^2\kappa^2)^2, \tag{2.13b}$$

and for brevity we have denoted

$$i\alpha c^{(n)} = -(a^{(n)} + ib^{(n)}), \tag{2.13c}$$

$$G = (D^2 - \kappa^2)\,\phi^{(1;\,1)}, \tag{2.13d}$$

$$H_{kn} = -(ma^{[n-m]} + ikb^{[n-m]})\,(D^2 - k^2\kappa^2)\,\phi^{\{k;\,m\}} + F_{kn}/(1 + \delta_{k0}), \tag{2.13e}$$

$$\begin{aligned} F_{kn} = &-(D\phi^{[k-j;\,n-m]})\,(ij[D^2 - j^2\kappa^2]\,\phi^{[j;\,m]}) - (D\tilde{\phi}^{[j;\,n-m]}) \\ &\times (i(k+j)\,[D^2 - (k+j)^2\kappa^2]\,\phi^{[k+j;\,m]}) - (D\phi^{[k+j;\,n-m]})\,(-ij[D^2 - j^2\kappa^2]\,\tilde{\phi}^{[j;\,m]}) \\ &+ (i(k-j)\,\phi^{[k-j;\,n-m]})\,(D[D^2 - j^2\kappa^2]\,\phi^{[j;\,m]}) + (-ij\tilde{\phi}^{[j;\,n-m]}) \\ &\times (D[D^2 - (k+j)^2\kappa^2]\,\phi^{[k+j;\,m]}) + (i(k+j)\,\phi^{[k+j;\,n-m]})\,(D[D^2 - j^2\kappa^2]\,\tilde{\phi}^{[j;\,m]}). \end{aligned} \tag{2.13f}$$

The boundary conditions become

$$\phi^{[k;\,n]} = D\phi^{[k;\,n]} = 0 \quad \text{at} \quad y = y_1, y_2. \tag{2.13g}$$

When these conditions are applied with $k = 0$, the total mean flow will be held constant, as we have specified.

Observing that the sums on the right-hand side of (2.13a) are empty for $k = n = 1$, we find the problem for $\phi^{(1;\,1)}$ as

$$\{(a^{(0)} + ib^{(0)} + i\alpha\bar{u})\,(D^2 - \kappa^2) - i\alpha(D^2\bar{u}) - R^{-1}(D^2 - \kappa^2)^2\}\,\phi^{(1;\,1)} = 0, \tag{2.14a}$$

$$\phi^{(1;\,1)} = D\phi^{(1;\,1)} = 0 \quad \text{at} \quad y = y_1, y_2. \tag{2.14b}$$

For two-dimensional motions, where $\kappa = \alpha$, (2.14) reduces to the Orr–Sommerfeld problem (1.2) with $\alpha c = -b^{(0)} + ia^{(0)}$. For three-dimensional motions, it is also the same eigenvalue problem as posed in a linear stability analysis. Thus, if $c(\alpha, \kappa, R) = c^{(0)}$ is the eigenvalue from the linear treatment,

$$b^{(0)} = -\alpha c_r \quad \text{and} \quad a^{(0)} = \alpha c_i. \tag{2.15}$$

The shape of the eigenfunction $\phi^{(1;\,1)}(y)$ is fixed by (2.14), but not its amplitude. If we define A in some particular manner, this definition will fix the amplitude of $\phi^{(1;\,1)}$, apart from a constant of modulus unity. An alternative approach is to arbitrarily normalize $\phi^{(1;1)}$ in some manner, and this will in turn define A implicitly. In the plane Poiseuille flow problem the latter choice is more convenient.

For plane Poiseuille flow, the mean velocity profile is†

$$\bar{u} = \tfrac{3}{2}(1-y^2). \tag{2.16}$$

The evenness of this profile and of the operators in (2.14*a*) permits separation of the eigenfunctions into a family of even and a family of odd functions (cf. Lin 1955). Following the normalization of Thomas (1953), we set

$$\phi^{(1;1)}(0) = 1 \quad \text{for even modes,} \tag{2.17a}$$

$$D\phi^{(1;1)} = 1 \quad \text{for odd modes.} \tag{2.17b}$$

With these normalizing conditions, it is only a matter of numerics to determine the eigenfunction and eigenvalue c.

Moving on to the higher-order problems, we see that (2.13) is sequentially solvable. With $\phi^{(1;1)}$ known, we can calculate the right-hand sides in the equations for $\phi^{(0;2)}$ and $\phi^{(2;2)}$, which then become inhomogeneous linear equations for these functions. We can continue the calculation, going next to $n = 3$, etc. provided that the constants $c^{(n)}$ can be determined in some appropriate manner along the way. We shall discuss this shortly.

At this point it seems desirable to make a comparison of the present expansion method with that of Stuart (1960*a*) and Watson (1960). In the present approach the constants $a^{(n)}$ and $b^{(n)}$ are real, and related directly to Watson's complex constants a_m, of which the most important is a_2. The highest harmonic contributing to terms of order A^n, denoted $\phi^{(n;n)}$ in the present paper, is proportional to the functions ψ_n used by Watson. The distortion of the mean velocity field, here described by the terms $\phi^{(0;n)}$, was represented by separate functions, f_n, in the Stuart–Watson formalism. Finally, whereas the present amplitude A is real, Stuart and Watson incorporate the temporal oscillation into their amplitude function A, and hence were led to working with complex amplitudes. A comparison between the two formalisms is given in table 1. An advantage of the present approach is that it involves fewer kinds of functions, and these are all described by equations of the same form. This permits a relatively simple computerization of the problem. The emergence of the perturbations in the wave speed, as reflected in the $b^{(n)}$, is somewhat simpler here. (Compare with the development leading to the last equation on p. 384 of Watson's 1960 paper.) It is interesting to see that the same formalism and virtually the same equations can be applied to *three*-dimensional disturbances of a particular yet important class.

We now turn to the question of evaluation of the $c^{(n)}$. The method which we shall employ for plane Poiseuille flow is essentially that of Stuart and Watson, which is appropriate when $|c_i|$ from the linearized treatment is small. An alternative method for the special case of equilibrium flows will also be suggested.

† The $\tfrac{3}{2}$ results from the normalization on the mixed mean velocity.

3. Evaluation of the $c^{(n)}$ when $|c_i|$ is small

Consider a general problem involving an nth order inhomogeneous linear differential equation, for which solutions are sought satisfying a set of n homogeneous boundary conditions; we represent this problem symbolically by

$$L_n \phi = \lambda f - g, \tag{3.1a}$$

$$\{\mathscr{L}_i \phi = 0\} \quad \text{at} \quad y_1 \quad \text{or} \quad y_2 \quad (i = 1, 2, \ldots, n). \tag{3.1b}$$

Stuart–Watson notation†	Present equivalence
$\lvert A \rvert$	A
$\bar{u}_\ell$	$\bar{u}$
a_n	$a^{(2n)} + ib^{(2n)}$
f_n	$\mathrm{Re}\{2\alpha^{-1} D\phi^{[0;2n]}\}$
ψ_n	$\alpha^{-1}\phi^{(n;n)}$
ψ_{nm}	$\alpha^{-1}\phi^{(n;2m+n)}$

† Stuart and Watson's equations are given for two-dimensional disturbances only.

TABLE 1. Comparison with the Stuart–Watson expansion.

As long as there are no non-trivial solutions to the associated homogeneous problem,

$$L_n w = 0, \tag{3.2a}$$

$$\{\mathscr{L}_i w = 0\} \quad \text{at} \quad y_1 \quad \text{or} \quad y_2 \quad (i = 1, 2, \ldots, n), \tag{3.2b}$$

the solution to (3.1) will exist and be unique. However, if eigensolutions to (3.2) exist, solutions to (3.1) can be found only for particular values of the parameter λ. These values may be determined with the aid of the adjoint function.

Suppose that it is possible to define an adjoint problem, where the adjoint differential operator L^* and the adjoint boundary conditions $\mathscr{L}^*$ are such that, if u and v are any two functions satisfying

$$\{\mathscr{L}_i u = 0\}, \quad \{\mathscr{L}_i^* v = 0\}, \quad (i = 1, 2, \ldots, n) \tag{3.3a, b}$$

(but not necessarily satisfying either of the differential equations), the operators are such that

$$\int_{y_1}^{y_2} u L_n^* v \, dy = \int_{y_1}^{y_2} v L_n u \, dy. \tag{3.3c}$$

The adjoint problem can be identified by integration by parts and will possess non-trivial solutions only if (3.2) does likewise.

Now, define an adjoint function Φ satisfying

$$L_n^* \Phi = 0, \quad \{\mathscr{L}_i^* \Phi = 0\} \quad \text{at} \quad y_1 \quad \text{or} \quad y_2 \quad (i = 1, 2, \ldots, n). \tag{3.4}$$

Then, multiplying (3.1a) by Φ, and integrating, using the fact that ϕ is to satisfy (3.1b), we have

$$\int_{y_1}^{y_2} \Phi L \phi \, dy = \int_{y_1}^{y_2} \phi L^* \Phi \, dy = 0 = \int_{y_1}^{y_2} \Phi(\lambda f - g) \, dy.$$

Of course, this is only meaningful if the adjoint exists, which will be the case only when (3.2) possesses eigensolutions. When this is the case, the parameter λ must be

$$\lambda = \int_{y_1}^{y_2} g\Phi\, dy \bigg/ \int_{y_1}^{y_2} f\Phi\, dy. \tag{3.5}$$

Only with this choice of λ can the solution to the inhomogeneous problem be found; but this solution is not unique, for we can always add a multiple of the eigensolution w.

Returning to (2.13a), we see that it is not likely that any of the associated homogeneous problems (other than that for $\phi^{(1;1)}$) will have non-trivial solutions, since for k and n other than 1 the operator L_{kn} differs from L_{11}. However, in the special case where $a^{(0)} = 0$, corresponding to $c_i = 0$, the operators L_{1n} are all identical with L_{11}, and the boundary conditions are identical, and consequently the associated homogeneous problem for $\phi^{[1;n]}$ would possess non-trivial solutions. Thus when c_i from the linear theory is *exactly* zero the higher-order constants $c^{[n]}$ *must* be determined from an equation of the form of (3.5). For points away from the linear neutral curve ($|c_i| \neq 0$) the constants $c^{[n]}$ could in principle be selected arbitrarily. However, practical difficulties would arise if c_i is very small, unless we retain the same scheme for picking the $c^{[n]}$. In order to make the solution continuous along a line passing through the neutral curve, it is probably desirable to employ this technique for all small $|c_i|$. A method which may lead to more rapid convergence when $|c_i|$ is not small is suggested in §4.

Integrating (2.14a) by parts, the adjoint problem can be developed as (Stuart 1960a)

$$\{i\alpha(\bar{u} - c^{(0)})(D^2 - \kappa^2) + 2i\alpha(D\bar{u})\, D - R^{-1}(D^2 - \kappa^2)^2\}\, \Phi = 0, \tag{3.6a}$$

$$\Phi = D\Phi = 0 \quad \text{at} \quad y = y_1, y_2. \tag{3.6b}$$

Then, once the adjoint has been found, the $c^{[n]}$ may be found from (see 2.13)

$$-i\alpha c^{[n-1]} = a^{[n]} + ib^{[n]} = \int_{y_1}^{y_2} H_{1n}\, \Phi\, dy \bigg/ \int_{y_1}^{y_2} G\Phi\, dy. \tag{3.7}$$

This can be done prior to the calculation of the harmonic contributions of order A^n. Except for the case $c_i = 0$ the $\phi^{(1;n)}$ will be unique. For the special case $c_i = 0$ the arbitrary multiple of $\phi^{(1;1)}$ can be varied by redefining the amplitude A.

An important observation is that the constants $c^{(n)}$ for odd n vanish, and thus are zero. By inspecting (2.13e) it may be seen that $H_{12} = 0$, and will be zero, except for the special case $c_i = 0$, in which $\phi^{(1;2)}$ could be a multiple of $\phi^{(1;1)}$; but we choose this multiple to be zero. Then, H_{03} and H_{23} both vanish, and hence these functions must also be zero. Following this line of inspection, we learn that the $c^{(n)}$ must vanish for odd n, and the functions H_{kn} and $\phi^{(k;n)}$ vanish if $k+n$ is odd. Thus, the remaining non-zero functions are those shown in table 2.

A physical interpretation of interactions leading to retention of these functions is also of interest. Since the non-linear interaction terms are quadratic, the fundamental interacts with itself to produce the zeroth and second harmonics,† both of order A^2. The $\phi^{(k;3)}$ terms come from the interaction of the $\phi^{(k;2)}$ terms with $\phi^{(1;1)}$, which produces a contribution to the fundamental of order A^3 as well

† By nth 'harmonic' we mean terms proportional to $\exp(in\theta)$.

as the third harmonic. The higher-order functions are formed in more complicated ways; for example, $\phi^{(2;4)}$ arises because of interaction of $\phi^{(1;1)}$ with $\phi^{(1;3)}$ and $\phi^{(3;3)}$, plus interaction of $\phi^{(0;2)}$ with $\phi^{(2;2)}$. In general, the function $\phi^{(k;n)}$ and the constant $c^{(n)}$ depend only on those functions $\phi^{(j;m)}$ for which $j+m \leqslant k+n$.

$$\psi = A^n[\phi^{(k;n)}e^{ik\theta} + \tilde{\phi}^{(k;n)}e^{-ik\theta}]$$

	$\phi^{(1;1)}$				
$\phi^{(0;2)}$		$\phi^{(2;2)}$			
	$\phi^{(1;3)}$		$\phi^{(3;3)}$		
$\phi^{(0;4)}$		$\phi^{(2;4)}$		$\phi^{(4;4)}$	
	$\phi^{(1;5)}$		$\phi^{(3;5)}$		$\phi^{(5;5)}$
	...				

Table 2. Non-zero functions.

A non-linear analysis when $c_i = 0$ is particularly of interest, for it will reveal whether disturbances which are neutrally stable if infinitesimal will grow or decay if their amplitude is finite. In the Poiseuille flow case to be considered shortly this is the question of prime concern. However, in Couette flow which appears to be stable to infinitesimal disturbances, the question arises as to whether or not finite amplitude motions might be unstable. Application of the expansion formalism in this type of problem requires a method for selection of the constants $c^{(n)}$ when $|c_i|$ is not small, and one such method will now be suggested.

4. Modified formulation for equilibrium flows

The special case of equilibrium flows ($dA/dt = 0$) can be handled in a somewhat different manner. While the expansion methods of the previous section in principle include equilibrium flows, a direct attack on these flows, in which dA/dt is assumed to be zero from the start, may lead to more rapidly convergent series, and in addition is conceptually somewhat more satisfying than the apparently arbitrary choosing of the $c^{(n)}$ which seems possible when $c_i \neq 0$.

Returning to (2.5), and setting $dA/dt = 0$, the resulting problem

$$\left[\omega + \frac{\partial\psi}{\partial y}\right]\frac{\partial\zeta}{\partial\theta} - \frac{\partial\psi}{\partial\theta}\frac{\partial\zeta}{\partial y} - \frac{1}{R}\left(\frac{\partial^2\zeta}{\partial y^2} + \kappa^2\frac{\partial^2\zeta}{\partial\theta^2}\right) = 0, \tag{4.1a}$$

$$\partial\psi/\partial\theta = \partial\psi/\partial y = 0 \quad \text{at} \quad y = \pm 1 \quad \text{(Poiseuille flow)}, \tag{4.1b}$$

can be viewed as a non-linear eigenvalue problem with ω as the eigenvalue. The harmonic expansion of ψ again leads to (2.7), with the dA/dt terms missing. To this point we presume that ω is real. However, in order to obtain a solution it is convenient to set a *mathematical* problem, based on (2.7), in which ω is allowed to be complex. This permits an expansion in powers of the amplitude, even though the *physical* problem (for real ω) may not have eigensolutions for continuously varying real A.

We then assume a power series expansion for ω,

$$\omega = \omega^{(0)} + A\omega^{(1)} + A^2\omega^{(2)} + \dots = A^n\omega^{(n)}, \tag{4.2}$$

where now the $\omega^{(n)}$ can be *complex*. If we again use (2.8) to expand the harmonic amplitudes, we obtain a set of equations similar to (2.13),

$$\mathscr{L}_{kn}\phi^{[k;n]} = -i\omega^{[n-1]}G\delta_{k1} + \mathscr{H}_{kn}, \tag{4.3a}$$

where the operator $\mathscr{L}_{kn}$ is now

$$\mathscr{L}_{kn} = ik[(\omega^{(0)} + \alpha\bar{u})(D^2 - k^2\kappa^2) - \alpha(D^2\bar{u})] - R^{-1}(D^2 - k^2\kappa^2)^2, \tag{4.3b}$$

and

$$\mathscr{H}_{kn} = -ik\omega^{[n-m]}(D^2 - k^2\kappa^2)\phi^{\{k;m\}} + F_{kn}/(1+\delta_{k0}), \tag{4.3c}$$

and F_{kn} and G are as defined by (2.13*d*) and (2.13*f*).

Note that the problem for $k = n = 1$ is again the Orr–Sommerfeld problem, with a *complex* $\omega^{(0)}$ replacing $-\alpha c$. This system can be solved sequentially in the manner discussed previously, provided that the constants $\omega^{(n)}$ can be suitably determined along the way. Now, since the operators $\mathscr{L}_{1n}$ are *all* identical with $\mathscr{L}_{11}$, the complementary equations for $\phi^{(1;n)}$ will all possess eigensolutions, and consequently the $\omega^{(n)}$ *must* be selected so that the adjoint orthogonality condition is satisfied. Hence, the $\omega^{(n)}$ *must* be chosen as

$$i\omega^{[n-1]} = \int_{y_1}^{y_2} \mathscr{H}_{1n}\Phi\,dy \Big/ \int_{y_1}^{y_2} G\Phi\,dy. \tag{4.4}$$

Having found all the $\omega^{(n)}$ for the mathematical problem in this manner, the solutions of the physical problem can be extracted by selecting the values of A for which ω is real. These are given by the roots of

$$A^n\omega_i^{(n)} = 0. \tag{4.5}$$

By examining the harmonic interactions, it can again be reasoned that the only contributing functions are those shown in table 2, and that the $\omega^{(n)}$ for odd n are zero. Hence determination of $\omega^{(2)}$ in this manner would allow estimation of the lowest eigenamplitude of the finite-amplitude equilibrium wave problem.

The modified formulation may be preferable for equilibrium, but it does not provide information relative to the stability of the flows. However, considering the phase plane diagrams (figure 1), we can argue that if $c_i < 0$ in the linearized analysis, so that the flow is stable to infinitesimal disturbances of the form in question, then the first equilibrium flow will be an *unstable* equilibrium, and the lowest root of (4.5) the *critical* amplitude. Conversely, if $c_i > 0$, so that infinitesimal disturbances grow in time, the first root of (4.5) would be the *equilibrium* amplitude. Hence, in problems where the equilibrium motion is of primary interest the alternative method can be used, and the meaning of the roots of (4.5) inferred in this manner.

5. Results for plane Poiseuille flow

The problem as formulated in §§2 and 3 have been applied to the case of plane Poiseuille flow. We consider the eigenmode which is unstable at the lowest Reynolds number, which appears to be the *even* mode treated asymptotically by

Lin (1955). The conditions at $y = -1$ may therefore be replaced by boundary conditions at $y = 0$. For the fundamental and adjoint these become

$$D\phi^{(1;1)} = D^3\phi^{(1;1)} = 0 \quad \text{at} \quad y = 0, \tag{5.1a}$$

$$D\Phi = D^3\Phi = 0 \quad \text{at} \quad y = 0. \tag{5.1b}$$

Now, considering the nature of the non-linear interaction terms, it is readily seen that the higher functions must be such that

$$\phi^{[k;n]} \quad \text{is} \quad \begin{cases} \text{odd for even } n, \\ \text{even for odd } n, \end{cases} \tag{5.1c}$$

and this provides the central boundary conditions for the higher-order functions.

The pertinent functions needed for evaluation of $c^{(2)}$ for two-dimensional disturbances have been computed numerically using the techniques described in appendix B, and the results are summarized in table 3. It is expected that the simplified single-precision methods described therein will be quite useful in related stability problems, where numerical difficulties brought about by the presence of a critical layer often force one to multiple precision programming.

R	α	c_r	c_i	$a^{(2)}$	$b^{(2)}$	k_1	k_2	k_3
6000	1·097	0·3773	0·0000	90·90	−206	−5·57	−5·71	193
5000	1·094	0·3896	0·0000	67·35	−172	−4·62	−3·97	143
3848·08†	1·02071	0·39603	0·00000	19·70	−111	−2·82	−1·79	44·0
5000	0·875	0·3516	0·0000	−3·268	−86·0	−2·13	−2·66	−1·74
6000	0·823	0·3304	0·0000	−7·088	−82·5	−2·04	−3·07	−9·06
4000	1·02071	0·3394	0·0005	19·89	−113	−2·72	−1·85	44·4
3500	1·02071	0·4026	−0·0015	22·02	−109	−3·49	−1·67	49·2

† Critical point.

TABLE 3. Summary of results for plane Poiseuille flow.

Of special interest are the results obtained at the critical point on the (linear) neutral stability curve. Here $a^{(2)}$ is positive ($+19·7$), indicating that disturbances actually *grow* at the critical Reynolds number for plane Poiseuille flow. The functions pertinent to this calculation are shown in figure 3.

Stuart (1960*a*) has enumerated the essential physical processes which contribute to $a^{(2)}$. He shows that $a^{(2)}$ can be decomposed into three parts, and written as

$$2a^{(2)} = k_1 + k_2 + k_3, \tag{5.2}$$

where k_1, k_2 and k_3 are as defined by Stuart's equations (6.3)–(6.5). Stuart observed that these three contributions arise from the following three physical processes: (1) the distortion of the mean motion (k_1), (2) the generation of the harmonic of the fundamental (k_2), (3) the distortion of the y-dependence of the fundamental (k_3). As Stuart observed, k_1 describes the change of the flow of energy to the disturbance due to distortion of the mean flow by the Reynolds stress, and is negative.† Stuart's (1958) integral method essentially considers

† For $|c_i|$ sufficiently small.

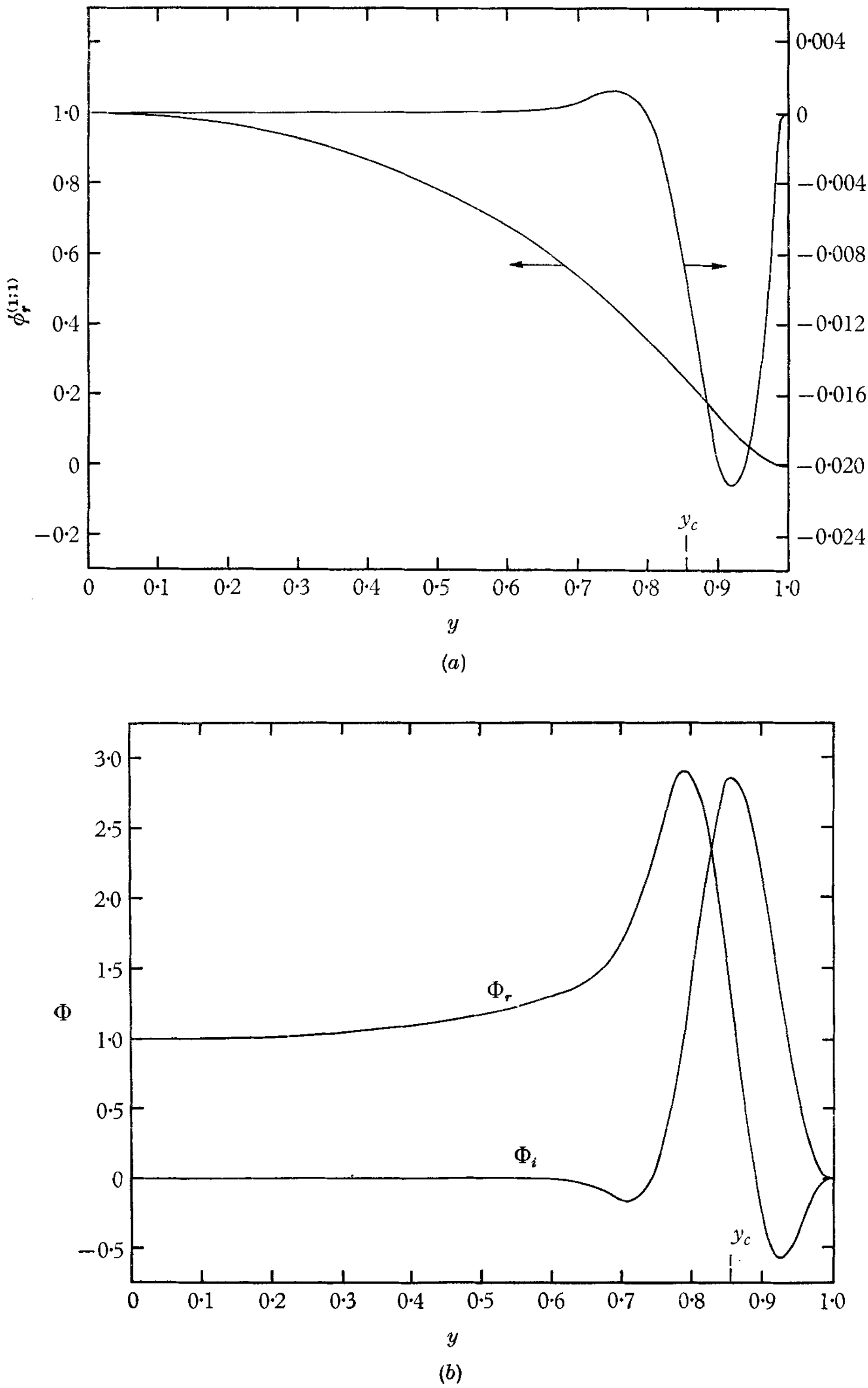
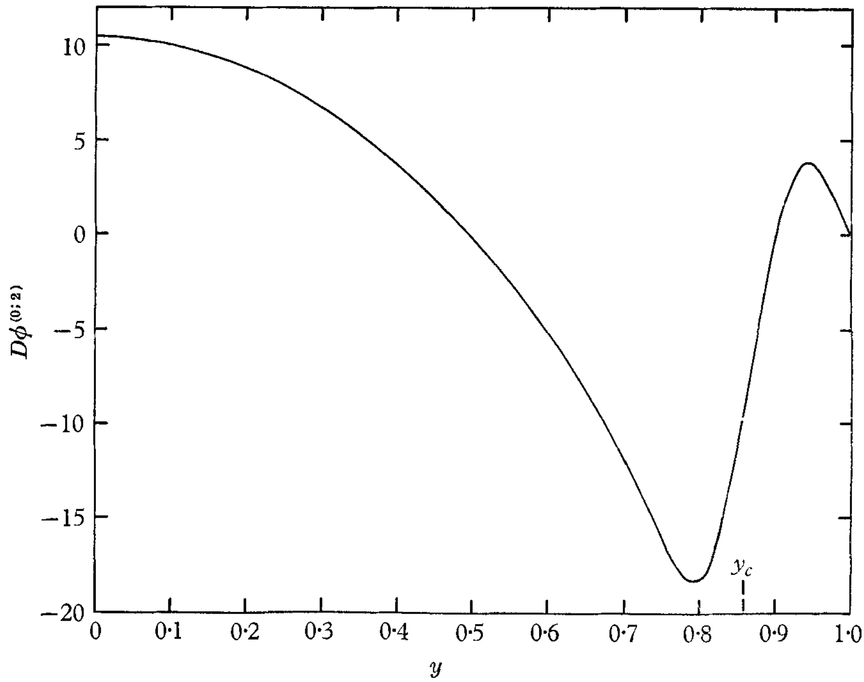
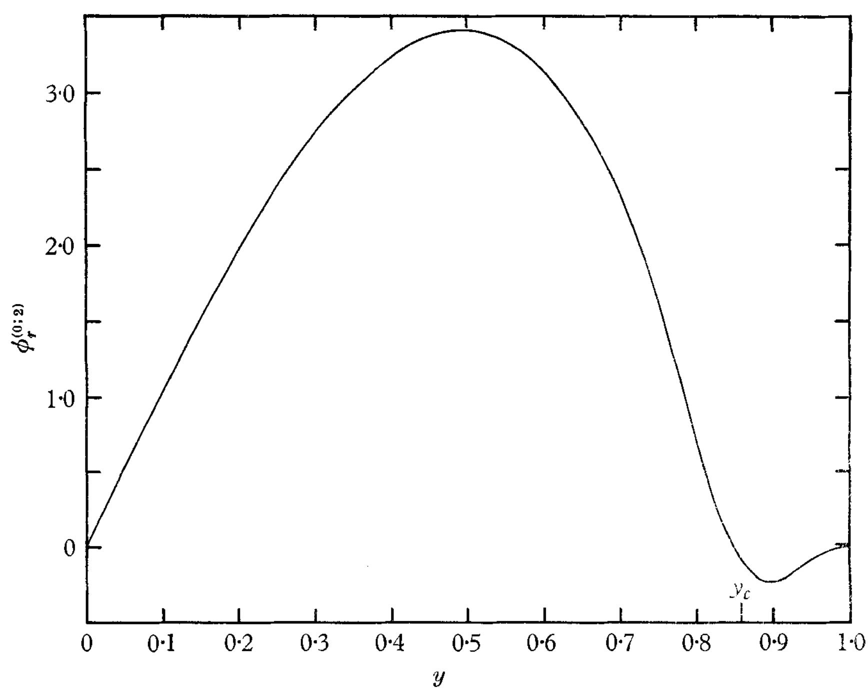


FIGURE 3*a*,*b*. For legend see p. 481.

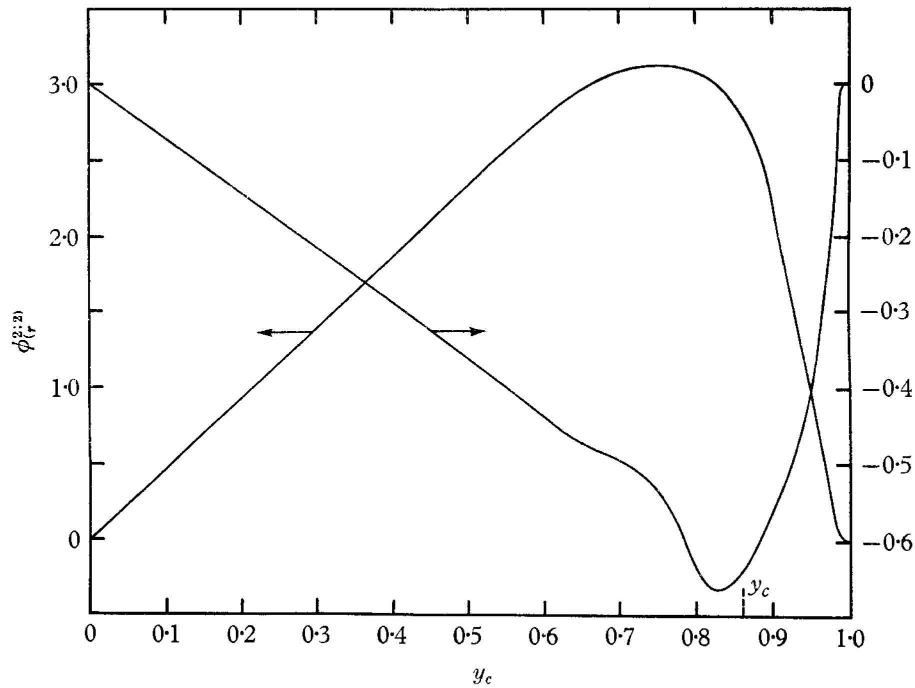


(*c*)



(*d*)

FIGURE 3*c*, *d*. For legend see p. 481.



(*e*)

FIGURE 3. Plane Poiseuille flow. $R = 3848{\cdot}08$, $\alpha = 1{\cdot}02071$, $c = 0{\cdot}396030$. (*a*) Eigenfunction at the critical point. (*b*) Adjoint eigenfunction at the critical point. (*c*) Mean velocity distortion at the critical point. (*d*) $\phi^{(0;2)}$ at the critical point. (*e*) $\phi^{(2;2)}$ at the critical point.

only k_1, and hence would always give $a^{(2)} < 0$, i.e. supercritical equilibrium flows. The flow of energy from the fundamental to the second harmonic is described by k_2, and Stuart (1960*a*) conjectured that it would be negative. The coefficient k_3 represents the modification of the energy of the fundamental due to distortion of its y-shape. The early analysis of Meksyn & Stuart (1951) included processes 1 and 3 approximately, but did not consider process 2. Stuart has conjectured that k_3 plays the dominant role in any flow which exhibits finite-amplitude subcritical instabilities.

Returning now to the present calculations, we see that k_1 and k_2 are indeed both negative, and that k_3 is large and positive at the critical point. Thus the distortion of the fundamental is indeed responsible for the subcritical instability and the absence of Taylor vortex-like finite-amplitude equilibrium motions in plane Poiseuille flow.

In the Stuart (1958) integral calculation, it was assumed that the disturbance retains its 'Orr–Sommerfeld' shape as its amplitude grows. The amplitude to which the disturbance grows was then determined by a balance of the fluctuation energy production and dissipation over the entire flow. The present results indicate that the distortion of the disturbance shape is quite important for this viscous type of instability. Davey's (1962) analysis of the growth of Taylor vortices in rotating Couette flow indicated that process 1 was the controlling factor; but the distortion of the fundamental was found to increase the amplitude of the equilibrium disturbance somewhat. It is now clear that one must consider the distortion of the fundamental in examining finite-amplitude motions resulting from viscous instability.

In the analysis of Meksyn & Stuart (1951) the mean velocity field was assumed to be distorted by a fluctuation having the shape of the Orr–Sommerfeld eigenfunction, and the stability of this modified mean field was examined by asymptotic methods. Hence they included the distortion of the mean and the fundamental approximately, but neglected the influence of the second harmonic. In order to assess the importance of $\phi^{(2;2)}$ a calculation was made in which this function was arbitrarily set to zero; the following values were obtained:

$$\left.\begin{aligned} R &= 3848\cdot08, \\ \alpha &= 1\cdot02071, \\ c &= 0\cdot39603, \end{aligned}\right\} \quad \begin{aligned} a^{(2)} &= 26\cdot1, \\ b^{(2)} &= -74\cdot3, \end{aligned} \quad \left\{\begin{aligned} k_1 &= -2\cdot82, \\ k_2 &= 0\cdot0, \\ k_3 &= +55\cdot0. \end{aligned}\right.$$

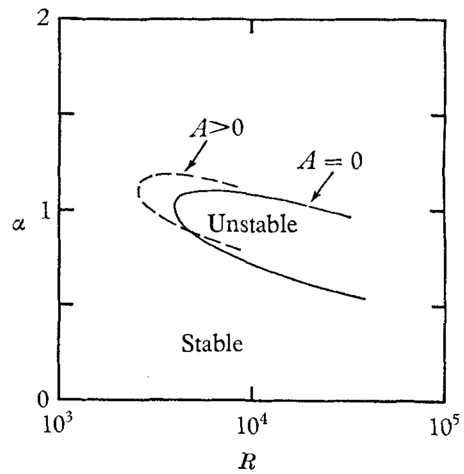


FIGURE 4. Qualitative amplitude dependence of the neutral stability curve.

Thus, total neglect of the second harmonic makes modestly significant changes in the results, with $a^{(2)}$ being increased by about 30 %, but the main result, that $a^{(2)}$ is positive, as indicated by the simpler analysis, is indeed correct. Davey (1962) found that the effect of the second harmonic was much less in the Taylor vortex case.

Calculations of $c^{(2)}$ have been carried out at four other points on the neutral stability curve, and reveal a surprising result. While $a^{(2)}$ appears to be positive on the upper branch, it is negative on the lower branch. Hence it would be possible to have finite-amplitude stable equilibrium motions if the disturbance could be kept very 'pure', i.e. if higher wave-number contributions could be suppressed, but it is doubtful that this could be done in a practical system.

Additional calculations were made at points off of the neutral curve. However, the differential equation for $\phi^{(0;2)}$ contains a term $2R\alpha^{(0)}D^2$, and this term (which was neglected in Stuart's (1960*a*) formulation, but not in Watson's (1960)) becomes quite important away from the neutral curve. Hence it is very important to know $a^{(0)}$ very accurately when it is not zero, and it is questionable whether this was the case in the present calculations. However, for the points reported in table 3, $a^{(0)}$ was sufficiently small and it is felt that the values of $a^{(2)}$ obtained are reasonably accurate. They show that $a^{(2)}$ is continuous and not rapidly changing across the neutral curve, and hence positive values for $a^{(2)}$ are expected both inside and outside of the neutral curve near the critical point. This further substantiates the conclusion that plane Poiseuille flow will exhibit subcritical instabilities, but no supercritical equilibrium flows.

We can estimate the reduction in the critical Reynolds number due to finite-amplitude disturbances. The value of the fluctuation intensity at the centre-line is, to order A,

$$(\overline{u_2'^2})^{\frac{1}{2}}/\bar{u}(0) = \tfrac{2}{3}\sqrt{2}|\phi^{(1;1)}|A.$$

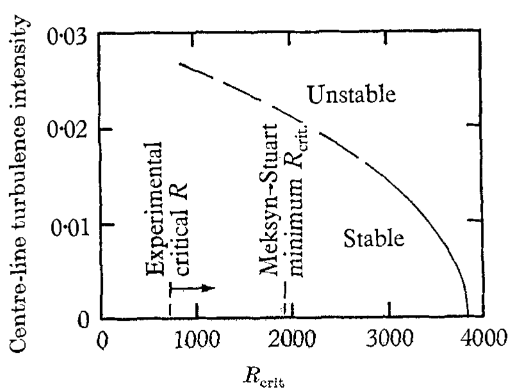


FIGURE 5. Estimated dependence of critical Reynolds number on turbulence intensity for plane Poiseuille flow.

Inspection of the eigenvalues indicates that $\partial a^{(0)}/\partial R$ in the vicinity of the critical point is about $0{\cdot}4 \times 10^{-5}$. Using $a^{(2)} = 20$, and setting $A^2 = -a^{(0)}/a^{(2)}$, and recalling the normalization of the eigenfunction, we have

$$(\overline{u_2'^2})^{\frac{1}{2}}/\bar{u}(0) \approx 0{\cdot}0005(3848 - R_{crit})^{\frac{1}{2}}. \tag{5.3}$$

This estimate of the critical Reynolds number is shown in figure 5. It appears that extremely weak disturbances can substantially reduce the critical Reynolds number. Davis & White (1928) found that the critical Reynolds number was often as low as 670; according to our estimate, this could be produced by a disturbance turbulence intensity of approximately 2·5 %, which is not unreasonable. The analysis of Meksyn & Stuart (1951) indicated a minimum critical Reynolds number of 1930, below which even large-amplitude disturbances could not upset the flow. The difficulty of making accurate calculations of $a^{(2)}$ off the neutral curve prevents a comparison with this result at the present time.

6. Results for Poiseuille–Couette flow

It is generally accepted that plane Couette flow is stable to infinitesimal disturbances at all Reynolds numbers, yet experimental evidence indicates that instability does exist for the large Reynolds numbers. It has thus been postulated that if finite amplitude disturbances were considered the resulting non-linear theory would predict instability. In order to apply the non-linear theory the linear problem must first be solved; thus the problem of plane Couette flow

u_w	R_{cr}	α_{cr}	c_{cr}	$a^{(2)}$	$b^{(2)}$	k_0	k_1	k_2	k_3
0	3,850	1·02	0·396	19·7	−111	1·97	−2·82	−1·79	44·3
0·0375	4,920	0·929	0·359	38·7	−124	4·44	−2·91	−3·72	84·0
0·075	8,380	0·800	0·288	59·5	−195	5·49	−3·95	−7·67	131
0·1125	11,200	0·700	0·236	26·8	−237	6·83	−5·06	−10·1	68·8
0·15	11,600	0·650	0·207	26·7	−258	7·82	−5·14	−12·0	70·6
0·225	10,900	0·569	0·162	77·2	−300	9·85	−5·00	−19·1	178
0·300	10,800	0·486	0·118	173	−347	13·2	−5·55	−38·7	390
0·375	11,900	0·380	0·0704	384	−348	20·4	−7·02	−110	886
0·45	15,900	0·270	0·0184	554	−6·3	40·5	−11·0	−286	1,410
0·48	20,200	0·216	−0·0058	459	124	62·8	−15·0	−344	1,280
0·495	24,500	0·181	−0·0185	376	158	89·4	−18·7	−376	1,150
0·51	33,600	0·136	−0·033	283	153	158	−26·4	−446	1,040
0·51375	37,600	0·122	−0·036	253	142	197	−29·8	−476	1,010
0·515625	40,800	0·113	−0·038	236	135	229	−32·4	−502	1,010
0·5175	44,300	0·104	−0·040	224	127	268	−35·3	−533	1,020
0·519375	48,800	0·0942	−0·041	204	115	329	−39·0	−569	1,020
0·52125	55,700	0·0839	−0·044	194	103	414	−44·7	−634	1,070
0·523125	65,100	0·0730	−0·047	187	87·6	547	−52·5	−725	1,150
0·5251	84,100	0·0560	−0·0479	168	60·9	925	−68·0	−903	1,310
0·526875	139,100	0·0324	−0·0501	178	17·4	2,500	−113	−1,447	1,915
0·527344	184,600	0·02577	−0·0505	203	−5·3	4,387	−150	−1,904	2,458
0·5275	213,640	0·02235	−0·051	223	−17·2	5,831	−173	−2,199	2,819
0·527844	500,700	0·00953	−0·0512	450	−90	32,000	−400	−5,100	6,400

TABLE 4. Results for combined Poiseuille–Couette flow.

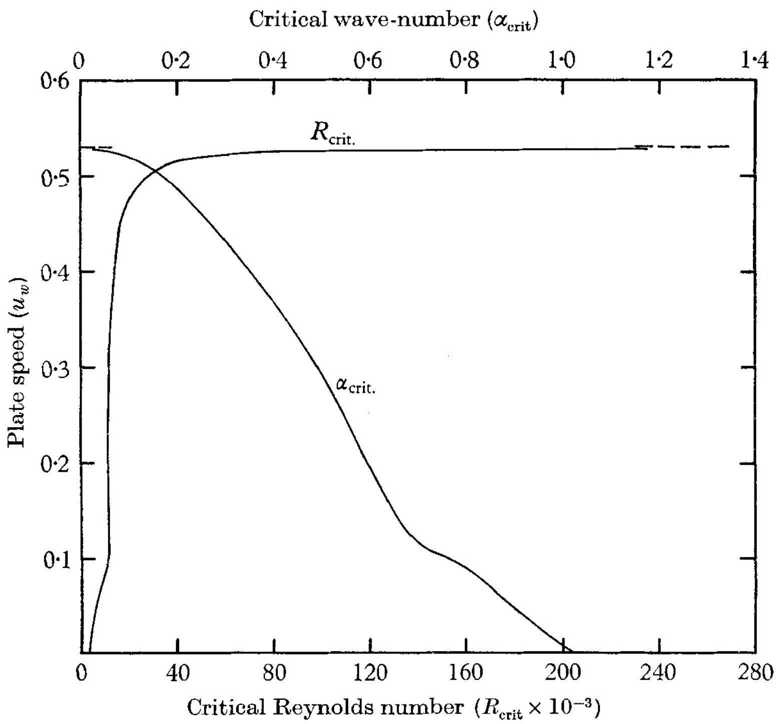


FIGURE 6. Critical points for combined Poiseuille–Couette flow.

remains, since the solutions of the linear problem have not yielded any neutrally stable modes. Potter (1966) has studied combinations of plane Poiseuille flow and Couette flow and has found that the linear theory yields stability at all the Reynolds numbers for flows in which the speed of the plates exceeds 53 % of the mean through-flow velocity. It is the intent of the present calculation also to approach Couette flow by starting with Poiseuille flow and adding to it components of Couette flow; then with the non-linear theory we can follow the effect of a finite (but small) disturbance as the flow becomes more and more stable to infinitesimal disturbances.

The differential equations leading to evaluation of $c^{(2)}$ have been solved numerically using an extension of the techniques described in appendix B.† The solution for $\phi^{(1;1)}$ requires iteration to determine the eigenvalue of the linearized

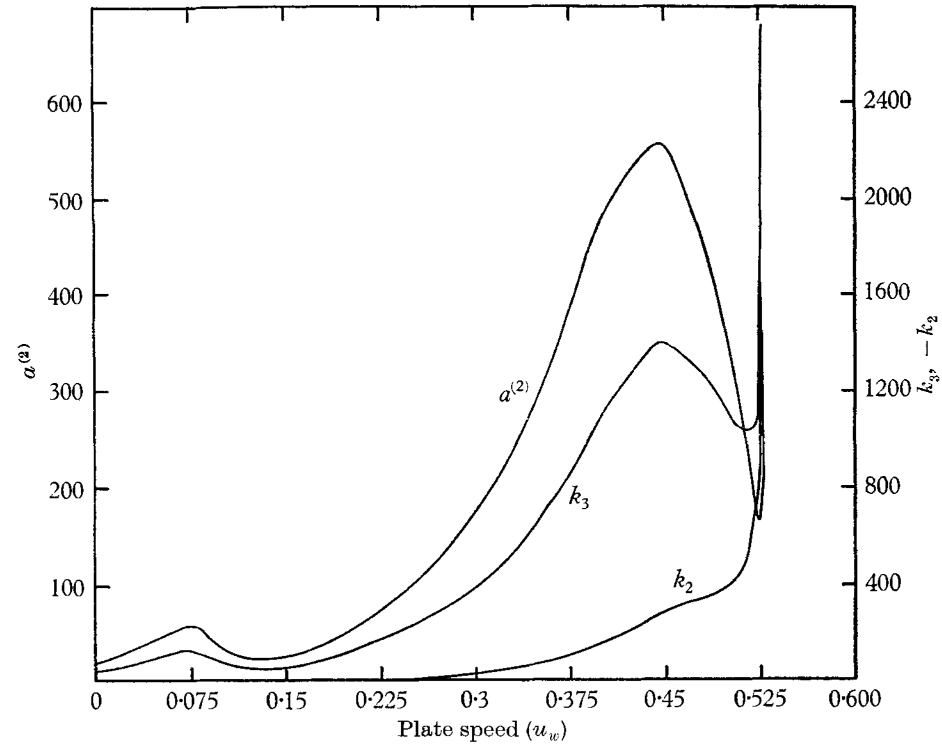


FIGURE 7. Non-linear constants at the critical points. Poiseuille–Couette flow.

stability problem, and the values determined by Potter (1966) using asymptotic methods provided useful first approximations. For each value of u_w one thereby obtains a neutral stability curve, and several were given by Potter. Here we confined our attention to the critical point at which the flow first becomes unstable to infinitesimal disturbances. Once the critical point had been determined for a given u_w, the calculations leading to evaluation of $c^{(2)}$ were completed; the results are summarized in table 4.

The location of the critical point is indicated as a function of u_w on figure 6. Note that the flow becomes stable to infinitesimal disturbances at all Reynolds numbers when u_w exceeds 0·528.‡ It is the value of $a^{(2)}$ for this case which is of chief interest; if it is positive, the flow is stable to infinitesimal disturbances, but

† It was necessary to integrate from both, walls and patch in the centre, since the eigenfunctions could not be split into even and odd modes.

‡ Potter (1966) obtained a value (based on different normalizations) of 0·700, using asymptotic methods. In his normalization the value determined 'exactly' would be 0·704.

unstable to finite disturbances. Indeed, this is the result obtained. A plot of $a^{(2)}$ *vs.* u_w is shown in figure 7. The calculations also indicate that k_3 dominates $a^{(2)}$ over most of the range, and $a^{(2)}$ is positive at the critical point over the entire range of u_w.

The eigenfunction $\phi^{(1;1)}(y)$ at the critical point for $u_w = 0{\cdot}526875$ is shown in figure 8. The 'outer viscous layers' and 'inner viscous layer' discussed by Lin (1955) are also shown. It should be noted that the thicknesses of these layers are

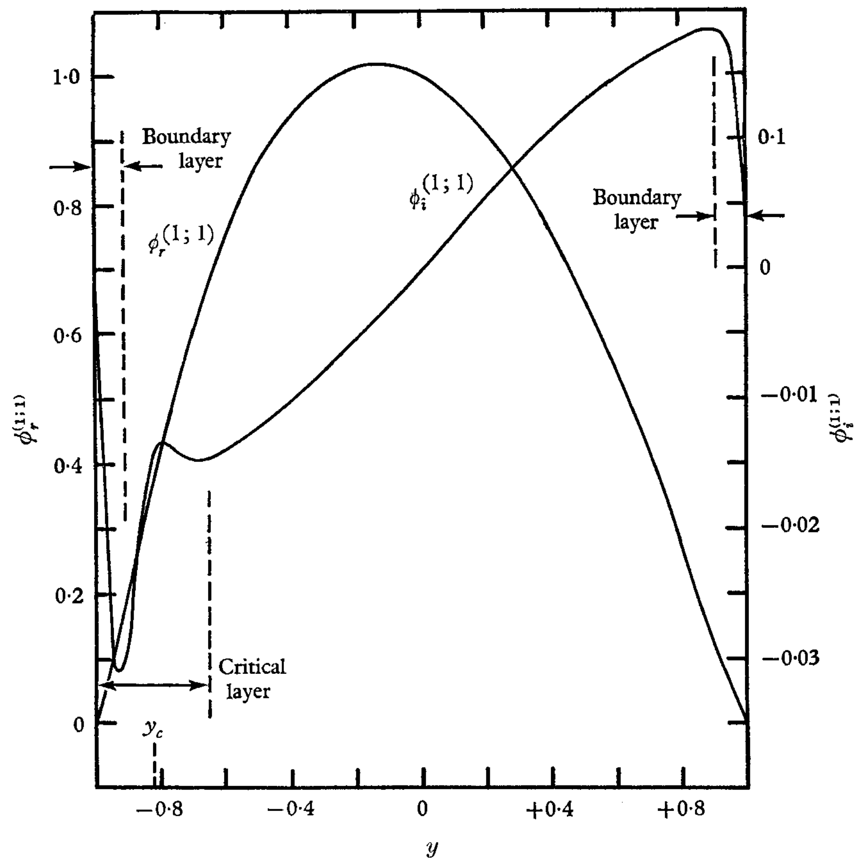


FIGURE 8. Critical eigenfunctions for $u_w = 0{\cdot}526875$. Poiseuille–Couette flow.

approximately the same as the thicknesses of those of figure 3*a*. This should be the case since, according to Lin, αR is the governing parameter and (αR) is approximately constant. The lowest order distortion of the mean, $\phi^{(0;2)}$, is shown in figure 9. Note that the net flow is unchanged, which is a constraint in the calculation.

These calculations clearly show that the flow remains unstable to finite disturbances, even when it is stable to infinitesimal disturbances, as has been frequently conjectured.

Most of the work reported in §§ 1–5 was done at the National Physical Laboratory, Teddington, England, where one of us (W.C.R.) was an NSF-sponsored Guest Scientist during 1964–65. The encouragement and most helpful suggestions which Dr J. T. Stuart and his associates provided during this period are sincerely appreciated. The Poiseuille–Couette problem calculations were carried out while M.C.P. attended an NSF RPCT program at Stanford during the summer of 1965. A grant from the Stanford Computation Centre made the main body of the calculations possible.

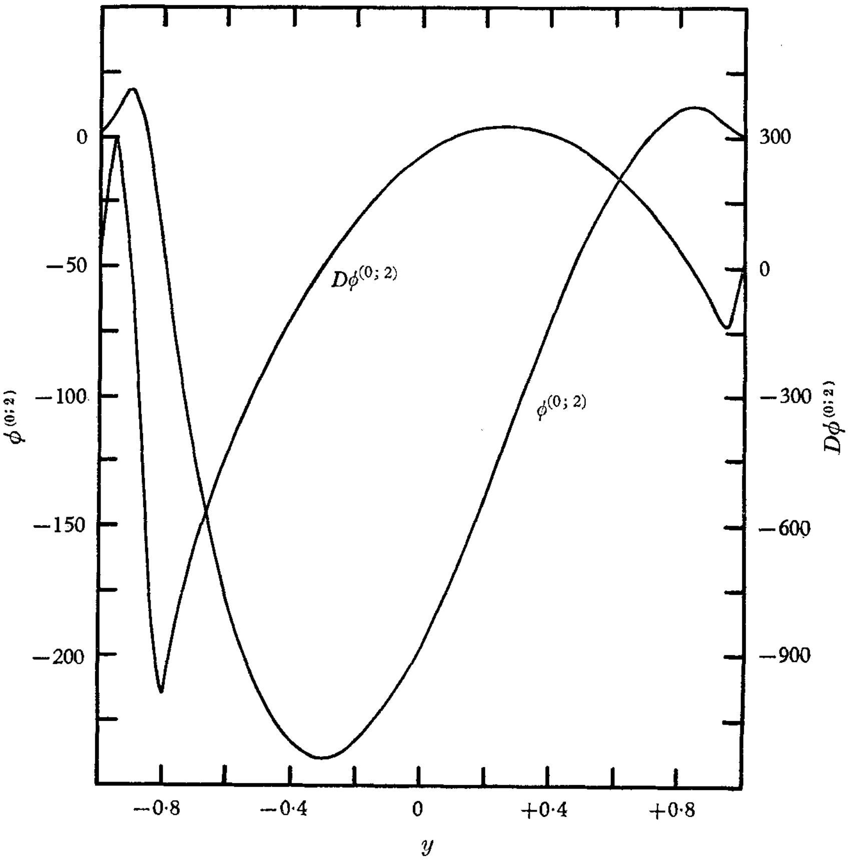


FIGURE 9. Mean distortion for $u_w = 0{\cdot}525875$. Critical eigenfunction, Poiseuille–Couette flow.

Appendix A. Superscript summation convention

A superscript summation convention has been found useful in compacting the expression of sums as well as manipulating, rearranging, and extracting terms. Broadly speaking, the convention is that the terms are summed over all possible (integer) values of repeated superscripts *or powers*, as in the familiar subscript summation convention used in manipulating with Cartesian tensors. Limits are placed on the values which an index can assume by punctuation, as indicated below:

$$(n) \quad n \geqslant 0,$$
$$[n] \quad n \geqslant 1,$$
$$\{n\} \quad n \geqslant 2,$$
$$n;m \quad n \leqslant m.$$

Negative indices are not permitted, and hence zero is by implication a lower bound. The delimiters ( ), [ ] and { } applied to a multiple superscript act only on the indices to the right of the semicolon:

$$[n;m] \quad \text{means} \quad 0 \leqslant n \leqslant m \geqslant 1.$$

Some examples which arise in non-linear problems are given below.

Superscript convention	Explicit
$\epsilon^n x^{(n)}$	$\sum_{n=0}^{\infty} \epsilon^n x^{(n)}$
$x^{(k-n)} y^{[n]}$	$\sum_{n=1}^{k} x^{(k-n)} y^{(n)}$
$\epsilon^k x^{[k-n]} y^{\{n\}}$	$\sum_{k=3}^{\infty} \sum_{n=2}^{k-1} \epsilon^k x^{(k-n)} y^{(n)}$
$x^{(j;\, n-m)} y^{(j;\, m)}$	$\sum_{m=0}^{n} \sum_{j=0}^{\min\binom{n-m}{m}} x^{(j;\, n-m)} y^{(j,\, m)}$
$x^{[k-j;\, n-m]} y^{[j;\, m]}$	$\sum_{m=1}^{n-1} \sum_{j=\max\binom{k-n+m}{0}}^{\min\binom{m}{k}} x^{(k-j;\, n-m)} y^{(j;\, m)}.$

Extraction of highest-order terms is particularly simple, for it merely requires change of punctuation:

$$\begin{aligned} x^{(k-n)} y^{(n)} &= x^{(k)} y^{(0)} + x^{(k-n)} y^{[n]} \\ &= x^{(k)} y^{(0)} + x^{(0)} y^{(k)} + x^{[k-n]} y^{[n]}. \end{aligned}$$

In non-linear problems one is often faced with the task of multiplying two expansions, and then grouping terms. The convention makes this quite easy:

$$\epsilon^n x^{(n)} \epsilon^m y^{(m)} = \epsilon^{n+m} x^{(n)} y^{(m)} = \epsilon^k x^{(k-m)} y^{(m)}.$$

We see that one merely has to introduce changes of notation in order to regroup the terms.

A particularly important case in the present work is the product of two Fourier expansions:

$$\begin{aligned} &[x^{(n)} e^{in\theta} + \tilde{x}^{(n)} e^{-in\theta}]\,[y^{(m)} e^{im\theta} + \tilde{y}^{(m)} e^{-im\theta}] \\ &\quad = e^{ik\theta} \left[x^{(k-m)} y^{(m)} + \frac{\tilde{x}^{(n)} y^{(k+n)} + x^{(k+m)} \tilde{y}^{(m)}}{1 + \delta_{k0}} \right] \\ &\qquad + e^{-ik\theta} \left[\tilde{x}^{(k-m)} \tilde{y}^{(m)} + \frac{x^{(n)} \tilde{y}^{(k+n)} + \tilde{x}^{(k+m)} y^{(m)}}{1 + \delta_{k0}} \right]. \end{aligned}$$

Appendix B. Numerical methods for the Poiseuille flow case

The required numerical calculations can be divided into two natural phases. First, the eigenvalues must be determined from a solution to the Orr–Sommerfeld problem. These values can then be used in subsequent numerical integration of the higher-order differential equations. There are some problems in numerically solving the ordinary differential equations when R is large, and the technique devised will very likely be useful in future calculations of this sort. The methods used for finding the eigenvalues are also of general interest.

(1) *Numerical solution of the differential equations*

We are faced with the task of solving linear ordinary differential equations of the form

$$L_i\phi + R^{-1}L_v\phi = f, \tag{B 1}$$

where L_i is a second-order operator, and L_v is a fourth-order operator. If R were of order unity there would be no real problem. However, in the present problem R is of the order of 10^4, and hence the equation is highly singular. There will be two linearly independent solutions of the (homogeneous) equation satisfying the two central boundary conditions, and an appropriate linear combination of these solutions, plus a particular solution of the inhomogeneous equation, must be combined to produce a solution satisfying the two wall boundary conditions. It is known that one of the solutions grows very rapidly away from the centre; a numerical calculation of this function for $R = 6666{\cdot}67$ (corresponding to Thomas's (1953) tabulated results), showed a growth of the order of 10^{18} from the centre to the wall. It is clear that generation of a second solution which is independent will be difficult; any slight round-off will in effect throw in some small multiple of the growing solution, which will likely dominate the second solution by the time the wall is reached. This was indeed observed in an experiment, in which an eigenfunction calculation was started at the centre with 'inviscid' starting conditions. By the time the wall was reached the solution was, to eight digits, a multiple of the growing solution, and in fact was of the order of 10^{10}, suggesting that initially it was present to the order of one part in 10^8, the maximum accuracy of the machine calculations!

Of course, the final solution does not exhibit this rapid growth, which means that only a very small amount of the growing homogeneous solution is required. The problem is to generate numerically a second homogeneous solution which is not merely a multiple of the growing homogeneous solution. There are essentially two approaches which come to mind. First, one might use multiple precision, extending the accuracy of the digital computations to, say, 16 digits. Then, the solution can be carried out in two pieces, outwards from the centre, and inwards from the wall, and matched somewhere between. This would limit the growth to the order of 10^8 on either side, for which 16 digit computations would suffice. A scheme of this type has been successfully employed by Nachtsheim (1964) in calculation of the eigenfunctions for plane Poiseuille flow. Using a fifth-order Milne predictor-corrector algorithm, and 256 steps, Nachtsheim was able to reproduce Thomas's (1953) eigenfunction very well. A second scheme, used by

Kaplan (1964), involves suppression of the growing solution during the calculation of the well-behaved solution. In the interests of using a scheme that would readily extend to very high Reynolds numbers, the suppression scheme of Kaplan was adopted.

In the present program the particular solution, the behaved homogeneous solution, and the growing homogeneous solution are simultaneously calculated, starting with appropriately chosen values at $y = 0$. The particular solution and the behaved homogeneous solution are started such that at $y = 0$ they represent solutions of 'inviscid' problems, and the growing solution is started with magnitude of the order of 10^{-8} of that for these solutions. If no steps were taken to suppress the growing solution from the behaved solutions, they would at first resemble solutions to the inviscid problems; however, before long the growing solution would begin to appear in these solutions, causing them to differ markedly from solutions of the inviscid equations. Hence at each step in the calculation a (small) multiple of the growing solution is subtracted from the behaved solutions; the multiple is chosen so as to continually adjust the behaved solutions so that they locally nullify the inviscid operators, i.e. $L_i(\phi) = 0$ for the behaved homogeneous solution.† In this manner the growing solution is prevented from ever dominating the behaved solutions. When the wall is reached we have in storage two linearly independent homogeneous solutions and a (behaved) particular solution. These are then combined to satisfy the wall boundary conditions.

The algorithm employed in the numerical integration is part of a predictor-corrector algorithm for fourth-order equations. However, the linearity of the equations eliminates the need for a predictor, and hence only the corrector equations are required. They are obtained by passing a third-order polynomial through ϕ'''' at four points, expressing the coefficients in terms of the (known) values of the fourth derivative at the three backward points and the single forward point. This is then integrated to give ϕ''' at the forward point in terms of the unknown fourth derivative, again to get ϕ'', and so forth. The resulting expressions for ϕ and its derivatives may be written as follows:

$$\left.\begin{aligned}
\phi_1 &= \phi_0+\phi_0'\Delta+\phi_0''\frac{\Delta^2}{2}+\phi_0'''\frac{\Delta^3}{6}+\frac{\Delta^4}{5040}(22\phi_1''''+214\phi_0''''-32\phi_{-1}''''+6\phi_{-2}''''),\\
\phi_1' &= \phi_0'+\phi_0''\Delta+\phi_0'''\frac{\Delta^2}{2}+\frac{\Delta^3}{720}(17\phi_1''''+120\phi_0''''-21\phi_{-1}''''+4\phi_{-2}''''),\\
\phi_1'' &= \phi_0''+\phi_0'''\Delta+\frac{\Delta^2}{360}(38\phi_1''''+171\phi_0''''-36\phi_{-1}''''+7\phi_{-2}''''),\\
\phi_1''' &= \phi_0'''+\frac{\Delta}{24}(9\phi_1''''+19\phi_0''''-5\phi_{-1}''''+\phi_{-2}'''').
\end{aligned}\right\} \quad \text{(B 2)}$$

Here Δ is the (uniform) mesh size and $\phi_n = \phi(y_n)$. Note that the last of (B 2) is the Adams corrector formula for a first-order equation. These expressions are substituted into the differential equation, yielding an equation for ϕ_1'''' of the form

$$A\phi_1''''+B+R^{-1}(C\phi_1''''+D) = F. \quad \text{(B 3)}$$

† A slightly different filter was used inside of the critical layer in the present calculations.

This is solved for ϕ_1''''. For very large R the inviscid terms A and B control the nature of the solution, and hence the scheme tends to be very stable. A similar scheme, based on a two-point fit, is used to start the calculation.

The eigenfunction given by Thomas (1953) at $R = 6666{\cdot}67$† formed a point o comparison in early calculation experiments. Using 401 net points, Thomas's eigenfunction could be reproduced to better than five decimal places. Subsequently the program was modified to employ a smaller mesh size within the critical layer, and it was found that equally satisfactory results could be obtained using $\Delta = 0{\cdot}005$ in the central region and $\Delta = 0{\cdot}0025$ within the critical layer. A calculation made using half as many points also seemed to be quite satisfactory as far as the eigenfunction was concerned, but small differences in the higher-order functions were evident.

The final calculations were made using $\Delta = 0{\cdot}005$ in the central region, and $\Delta = 0{\cdot}0025$ within the critical layer. Values of the functions were retained at 201 equally spaced points (corresponding to $\Delta = 0{\cdot}005$) in the range $0 \leqslant y \leqslant 1$ for use in the higher-order calculations.

The heart of the program was a subroutine for integrating a generalized inhomogeneous Orr–Sommerfeld equation or its homogeneous adjoint equation. A second subroutine provides for automatic collection of terms and calculation of a generalized H_{kn}. Given the eigenvalue, repeated calls of these subroutines, plus numerical integration, were all that were required. Integrals were calculated using a five-point scheme. All programming was done in FORTRAN-IV, which provides for automatic complex arithmetic, and the calculations were executed on an IBM 7090. Slightly over one minute is required for computation of $a^{(2)}$ and the associated functions if the eigenvalue is initially known.

Although the functions $\phi^{(1;1)}$, $\phi^{(0;2)}$ and $\phi^{(2;2)}$, and the adjoint, could be calculated in this manner, the accuracy in $\phi^{(1;3)}$ was relatively poor. This is because the inhomogeneous term H_{13} is highly oscillatory, especially within the critical layer, and the 201 points at which values obtained were evidently insufficient to allow accurate calculation of $\phi^{(1;3)}$. It might be thought that the calculation of $c^{(2)}$ would be plagued by this same difficulty, but this did not appear to be the case. Calculations carried out with 101 retention points, using the same integration steps as in the 201 point case, give a $c^{(2)}$ that was only a few per cent different from that obtained in the 201 point case. Doubling the number of integration steps, maintaining 201 retention points, made less than a $0{\cdot}1\,\%$ change in $c^{(2)}$. Hence the accuracies of $c^{(2)}$ and the functions involved in its evaluation are considered quite satisfactory. Since $\phi^{(1;3)}$ could not be calculated with sufficient accuracy, k_3 was obtained by subtraction using (5.2).

(2) *Eigenvalue determination*

An iterative scheme was developed for determination of the eigenvalues, using the differential equation integrator described above. The scheme has several options, listed below: (1) find c for R,α fixed, (2) find R,α for c fixed ($c_i = 0$),

† Note that R in Thomas's paper is $\frac{3}{2}$ that in the present work, his c is $\frac{3}{2}$ times the conjugate of our c, and his ϕ corresponds to $\tilde{\phi}$ here, owing to differences in normalization and harmonic representation.

(3) find α, c_r for R fixed ($c_i = 0$). Options (2) and (3) are useful in establishing the neutral curve, and employ an interesting dual adjustment scheme, which we now describe.

Using the equation integrating scheme, we can construct a solution for given α, R and c, which satisfies the central boundary conditions and *one* of the wall conditions. The game is to adjust the parameters so that *both* wall conditions are satisfied. Let T denote the 'test function', which should be zero. In the present case this was chosen as $D\phi^{(1;1)}(1)$. For option (2), $T = T(\alpha, c_r)$. With three successive equation solutions in the vicinity of α, c_r, we establish $\partial T/\partial\alpha$ and $\partial T/\partial c_r$. Then, the value of T at the new trial point is approximated by

$$0 = T(\alpha+\Delta\alpha,\, c_r+\Delta c_r) = T_0 + (\partial T/\partial\alpha)_0\Delta\alpha + (\partial T/\partial c_r)_0\Delta c_r.$$

This is used to correct α and R.

When the asymptotic results of Lin (1955) were used as starting values, satisfactory convergence was obtained in three or four passes, and the eigenvalue could be established in slightly more than 1 minute. A similar scheme was used in option (3), and a direct Newton method employed for option (1). The critical point was calculated using option (2), holding c_r at the value given by Nachtsheim (1964). This slight trimming was necessary because the test function is quite sensitive to the values of c, α and R, and Nachtsheim reported only four significant figures.

REFERENCES

Coles, D. 1965 *J. Fluid Mech.* **21**, 385.

Davey, A. 1962 *J. Fluid Mech.* **14**, 336.

Davies, S. J. & White, C. M. 1928 *Proc. Roy. Soc. Lond.*, A **119**, 92.

DiPrima, R. C. & Stuart, J. T. 1964 Nonlinear aspects of instability in flow between rotating cylinders. *NPL Aero. Report* 1111. (Presented at XIth Int. Congr. App. Mech., Munich 1964.)

Kaplan, R. E. 1964 Solution of Orr–Sommerfeld equation for laminar boundary layer flow over compliant boundaries. ASRL-TR-116-1, *Cambridge Aeroelastic and Structures Research Lab. Report, Mass. Inst. Tech.*, June 1964 (STAR N64-29052).

Lin, C. C. 1955 *The Theory of Hydrodynamic Stability*. Cambridge University Press.

Meksyn, D. & Stuart, J. T. 1951 *Proc. Roy. Soc.* A **208**, 517.

Nachtsheim, P. R. 1964 An initial value method for the numerical treatment of the Orr–Sommerfeld equation for the case of plane Poiseuille flow. *NASA TN D*-2414.

Potter, M. C. 1966 *J. Fluid Mech.* **24**, 609.

Stuart, J. T. 1958 *J. Fluid Mech.* **4**, 1.

Stuart, J. T. 1960*a* *J. Fluid Mech.* **9**, 353.

Stuart, J. T. 1960*b* Non-linear effects in hydrodynamic stability. *Proc. Xth Int. Cong. App. Mech.* Stresa, 1960. Elsevier, 1962.

Stuart, J. T. 1961 On three-dimensional non-linear effects in the stability of parallel flows. *Adv. Aero. Sci.* **3–4**, 121. Pergamon Press.

Thomas, L. H. 1953 *Phys. Rev.* **91**, 780.

Watson, J. 1960 *J. Fluid Mech.* **9**, 371.

Watson, J. 1962 *J. Fluid Mech.* **14**, 221.